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Modeling and analysis of a long thin conducting stripline

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Abstract. A long thin conducting stripline embedded in a dielectric and centered between two large conducting plates, *i.e.*, the stripline environment, is considered. The stripline is modeled as infinitely long, infinitely thin, and perfectly conducting by first considering a stripline of finite length, thickness, and conductivity in a dielectric layer. Starting from Maxwell's equations and assuming that the current on the stripline is a propagating wave in length direction, asymptotic expressions for the fields inside and in the neighbourhood of the stripline are deduced. These expressions are used to model the stripline in the stripline environment, which leads to a boundary-value problem for the electric potential. This problem is solved by two different approaches, leading to integral equations for the current and for an auxiliary function describing the electric potential. A relation between the current and the auxiliary function is deduced, which is used to obtain asymptotic expressions for current and impedance. Results are compared with a numerical solution of the integral equation for the current and with results in literature.

Key words: current distribution, impedance, perfect conduction, skin depth, stripline

1. Introduction

A stripline is a special type of electromagnetic transmission line that is used for the excitation of antennas. There exists a number of different stripline configurations, one of which is considered in this paper; see Figure 1. Other examples are two or more strips aside or above one another, embedded in a dielectric and positioned between two ground plates or in a rectangular waveguide; see Collin [1, p. 170]. For high-frequency applications, a stripline is modeled usually as infinitely long, infinitely thin, and perfectly conducting, because the skin depth, as defined by Landau and Lifshitz [2, Chapter 7], is much smaller than the thickness of the stripline, and the thickness is much smaller than all other characteristic length scales. By this model, the electromagnetic fields of a number of planar striplines can be computed by means of conformal-mapping theory, if a transverse electromagnetic wave (TEM) is assumed. For one or two strips between two ground plates, examples are given in Collin [1, Chapter 3 (Part 2) and Appendix 3] and Wheeler [3]. For a planar array of strips contained in a rectangular waveguide, such an example is given in Homentcovschi *et al.* [4]. To calculate the electromagnetic field of more complicated stripline configurations with several dielectric layers and strips positioned at different heights, numerical techniques are used. Examples of such techniques are the partial boundary-element method (see Atsuki and Li [5]) and the spectral-domain immitance approach; see Huang and Tzuang [6]. For a comparison of different numerical techniques, we refer to the introductions of these two papers and the references cited therein. Recently, striplines with non-uniform strip widths were studied by Chiu [7] using a quasi-static approximation for the capacitance. Moreover, both perfectly conducting and superconducting striplines were studied by Bonnet-Ben Dhia and Ramdani

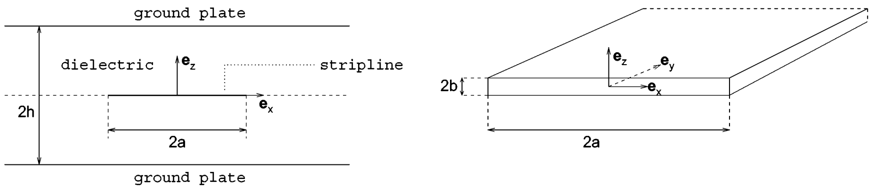

Figure 1. Left: Cross-section of a stripline in a stripline environment, Right: Stripline of finite thickness.

[8], who proved the existence of guided modes. The striplines in the latter two papers are not inside a waveguide or between two ground plates, but on a dielectric substrate located on a ground plate. Hence, the striplines radiate into space.

In this paper, we are interested in the modeling and analysis of both a non-perfectly and a perfectly conducting long thin stripline as shown in Figure 1. For the non-perfectly conducting long thin stripline, we wish to deduce asymptotic expressions for the fields in the neighbourhood of the strip. Such expressions can for example be used for an estimate of the near field of the strip. For the perfectly conducting stripline, we wish to use an integral-equation technique instead of a conformal-mapping technique or a numerical technique. Moreover, we wish to compare results obtained for the impedance of the stripline with empiric formulas in literature. The analyses of the non-perfectly conducting and the perfectly conducting stripline are linked by showing by dimensional analysis that the asymptotic expressions for the former amount to the boundary conditions for the latter.

Figure 1 shows the cross-section of a stripline in a stripline environment and the geometry of the stripline itself. The stripline is a long thin conducting strip of width $2a$ and thickness $2b$ with $b \ll a$. In the stripline environment, the stripline is centered between two conducting plates, called ground plates, which are connected to the earth. The layer between the groundplates and around the stripline is filled by a dielectric medium. The width of the ground plates and the dielectric layer is much larger than the width of the stripline, while their length equals that of the stripline. Moreover, the ground plates are separated by a distance $2h$, which is usually of the same order as the width $2a$, *i.e.*, $\zeta = h/a$ is of order 1. For the stripline we assume the existence of an electric current as a propagating wave, with prescribed total amplitude I. Hence, the total time-averaged current is prescribed. We assume that the wavelength λ of this wave is much smaller than the length of the stripline, while $2a \ll \lambda$. Therefore, we may assume that the current propagates in length direction (y-direction) only. We disregard reflections and boundary effects at the ends of the stripline. The thus defined problem is symmetric in z. Reference parameter values for a copper stripline designed for high frequency applications are given in Table 1.

In Section 2, we consider a long thin conducting stripline in a dielectric layer. Starting from Maxwell's equations and assuming that the current on the stripline is a propagating wave in y-direction, we deduce asymptotic expressions for the fields inside and in the neighbourhood of the stripline. We show under which conditions these expressions can be applied. Moreover, we show that these expressions and conditions imply that the stripline can be modeled as infinitely long, infinitely thin, and perfectly conducting. Using the obtained results, we model a stripline in a stripline environment in Section 3. As in the literature, we deduce a boundary-value problem for the electric potential. In Section 4, we solve this boundary-value problem by two different approaches, one leading to an integral equation for the current and the other to an integral equation for an auxiliary function describing the electric potential. A relation between the current and the auxiliary function is deduced, which is used to obtain asymptotic

Table 1. Parameter values for a stripline. The subscripts 0, d, s refer to vacuum, dielectric, and stripline, respectively.

Frequency	f	1–40 GHz
Radian frequency	$\omega = 2\pi f$	$2\pi - 80\pi$ GHz
Permittivity	ε_0	$1/36\pi \times 10^{-9}$ A s/V m
	$\varepsilon_n/\varepsilon_0$ $(n = d, s)$	1–100
Permeability	$\mu_0 = \mu_d = \mu_s$	$4\pi \times 10^{-7}$ V s/A m
Speed of light	$c_0 = 1/\sqrt{\varepsilon_0\mu_0}$	3×10^8 m/s
	$c_n/c_0 = \sqrt{\varepsilon_0/\varepsilon_n}$, $(n = d, s)$	
Wavelength	$\lambda_0 = c_0/f$	0·3 m
	$\lambda_n/\lambda_0 = c_n/c_0$ $(n = d, s)$	
Wave number	$k_0 = 2\pi/\lambda_0 = \omega\sqrt{\varepsilon_0\mu_0}$	20·9 m^{-1}
	$k_n/k_0 = \lambda_0/\lambda_n = \sqrt{\varepsilon_n/\varepsilon_0}$ $(n = d, s)$	
Width	$2a$	10^{-2} m
Thickness	$2b$	40 μm
Conductivity	σ	$5{\cdot}9 \times 10^7$ $\mathrm{ohm}^{-1}\mathrm{m}^{-1}$
Height	h	10^{-2} m

expressions for current and impedance in Section 5. In Section 6, a numerical algorithm is given to solve the integral equation for the current. Results of this algorithm for current and impedance are shown in Section 7. Finally, the numerical solution of the impedance is compared to the asymptotic solutions and to results from the literature.

2. Modeling a long thin conducting stripline

In this section, we consider a long thin conducting stripline in a dielectric layer, as described in the previous section. We will deduce asymptotic expressions for the fields inside and in the neighbourhood of the stripline. We assume that the two media, stripline and dielectric, are linear, homogeneous, and isotropic. Furthermore, the permeability of both media equals the permeability of vacuum, $\mu_s = \mu_d = \mu_0$, while their permittivities ε_d and ε_s are equal to, or at least of the same order as the permittivity ε_0 of vacuum; see Table 1. The electric field $\boldsymbol{\mathcal{E}}^{(d)}$ in the dielectric is described by the wave equation and the condition that it is solenoidal,

$$\frac{\partial^2 \boldsymbol{\mathcal{E}}^{(d)}}{\partial t^2} - c_d^2 \Delta \boldsymbol{\mathcal{E}}^{(d)} = \mathbf{0}, \qquad \operatorname{div} \boldsymbol{\mathcal{E}}^{(d)} = 0, \tag{1}$$

where $c_d^2 = 1/(\mu_0\varepsilon_d)$. These equations are deduced from Maxwell's equations. The magnetic field $\boldsymbol{\mathcal{H}}^{(d)}$ can be calculated from Maxwell's equations, when the electric field is known. We note that the current density $\boldsymbol{\mathcal{J}}^{(d)} = \mathbf{0}$ and the free-charge density $\varrho^{(d)} = 0$.

In the stripline, no free-charge distribution can exist, *i.e.*, $\varrho^{(s)} = 0$, and $\boldsymbol{\mathcal{J}}^{(s)} = \sigma \boldsymbol{\mathcal{E}}^{(s)}$ by Ohm's law. Then, $\boldsymbol{\mathcal{J}}^{(s)}$ is described by the damped-wave equation and the condition that it is solenoidal,

$$\frac{\partial^2 \boldsymbol{\mathcal{J}}^{(s)}}{\partial t^2} + \frac{\sigma}{\varepsilon_s}\frac{\partial \boldsymbol{\mathcal{J}}^{(s)}}{\partial t} = c_s^2 \Delta \boldsymbol{\mathcal{J}}^{(s)}, \qquad \operatorname{div} \boldsymbol{\mathcal{J}}^{(s)} = 0. \tag{2}$$

Furthermore, $\boldsymbol{\mathcal{E}}^{(s)}$ and $\boldsymbol{\mathcal{H}}^{(s)}$ follow from Maxwell's equations and Ohm's law, when $\boldsymbol{\mathcal{J}}^{(s)}$ is known.

The transition conditions for the field quantities $\boldsymbol{\mathcal{E}}^{(d)}$ and $\boldsymbol{\mathcal{J}}^{(s)}$ at the interfaces $x = \pm a$ and $z = \pm b$ follow from the general transition conditions, see Stratton [9, pp. 34–38]. We consider only the interfaces $z = \pm b$ for reasons that will be explained later. The normal component of the magnetic field and the tangential component of the electric field are continuous across these interfaces, yielding

$$\left(\operatorname{rot}\boldsymbol{\mathcal{E}}^{(d)}\right)_z - \frac{1}{\sigma}\left(\operatorname{rot}\boldsymbol{\mathcal{J}}^{(s)}\right)_z \Big|_{z=\pm b} = 0, \tag{3}$$

$$\mathcal{E}_y^{(d)} - \frac{1}{\sigma}\mathcal{J}_y^{(s)} \Big|_{z=\pm b} = 0, \qquad \mathcal{E}_x^{(d)} - \frac{1}{\sigma}\mathcal{J}_x^{(s)} \Big|_{z=\pm b} = 0. \tag{4}$$

It can be shown that (4) implies (3).

The tangential component of the magnetic field is continuous, because the (volume) current $\mathcal{J}^{(s)}$ is finite at the interface. Hence,

$$\left(\operatorname{rot}\boldsymbol{\mathcal{E}}^{(d)}\right)_y - \frac{1}{\sigma}\left(\operatorname{rot}\boldsymbol{\mathcal{J}}^{(s)}\right)_y \Big|_{z=\pm b} = 0, \quad \left(\operatorname{rot}\boldsymbol{\mathcal{E}}^{(d)}\right)_x - \frac{1}{\sigma}\left(\operatorname{rot}\boldsymbol{\mathcal{J}}^{(s)}\right)_x \Big|_{z=\pm b} = 0. \tag{5}$$

The normal component of the dielectric displacement is discontinuous across the interfaces $z = \pm b$, as surface charges exist at these interfaces. Let $\varrho_S^{\pm}$ be the (unknown) surface charge density at $z = \pm b$. Because of symmetry, $\varrho_S^+ = \varrho_S^- =: \varrho_S$. Then,

$$\varrho_S = \pm\varepsilon_d \mathcal{E}_z^{(d)}\big|_{z=\pm b}\,. \tag{6}$$

Here we use that the normal component of the current is continuous across the the interfaces $z = \pm b$,

$$\mathcal{J}_z^{(s)}\big|_{z=\pm b} = 0. \tag{7}$$

As mentioned before, we assume that the current in the strip is a propagating wave in the y-direction,

$$\boldsymbol{\mathcal{J}}^{(s)}(\mathbf{x}, t) = \mathbf{J}^{(s)}(x, z)\, \mathrm{e}^{\mathrm{i}(\beta y - \omega t)}. \tag{8}$$

where β is the wave number with $\mathfrak{Re}\,\beta > 0$. It should be noted that this wave number is unknown and its determination is an important part of the analyses to follow. Note that for $\mathfrak{Re}\,\beta = 0$ the wave does not propagate, and for $\mathfrak{Re}\,\beta < 0$ the wave propagates in the negative y-direction. Moreover, depending on the sign of $\mathfrak{Im}\,\beta$, the wave is attenuated, explodes, or is purely propagating in the positive y-direction.

By the transition conditions, $\boldsymbol{\mathcal{E}}^{(d)}$ and ϱ_S are of the same nature,

$$\boldsymbol{\mathcal{E}}^{(d)}(\mathbf{x}, t) = \mathbf{E}^{(d)}(x, z)\, \mathrm{e}^{\mathrm{i}(\beta y - \omega t)}, \quad \varrho_S(x, y, t) = \rho_S(x)\, \mathrm{e}^{\mathrm{i}(\beta y - \omega t)}. \tag{9}$$

We introduce the dimensionless coordinates $\hat{x} = x/a$ and $\hat{z} = z/b$, and the parameters

$$\epsilon = \frac{b}{a}, \qquad \xi^2 = b^2(\beta^2 - k_d^2), \qquad \frac{1}{\delta^2} = b^2(\beta^2 - k_s^2 - \mathrm{i}\sigma\omega\mu_0), \tag{10}$$

with $-\pi/2 < \arg\xi \leq \pi/2$ and $-\pi/2 < \arg\delta \leq \pi/2$. Then, the field equations (1) and (2) turn into (we omit the superscripts)

$$\epsilon^2 \frac{\partial^2 \mathbf{E}}{\partial \hat{x}^2} - \xi^2 \mathbf{E} + \frac{\partial^2 \mathbf{E}}{\partial \hat{z}^2} = \mathbf{0}, \qquad \epsilon \frac{\partial E_x}{\partial \hat{x}} + \mathrm{i}b\beta E_y + \frac{\partial E_z}{\partial \hat{z}} = 0, \tag{11}$$

$$\epsilon^2 \frac{\partial^2 \mathbf{J}}{\partial \hat{x}^2} - \frac{1}{\delta^2} \mathbf{J} + \frac{\partial^2 \mathbf{J}}{\partial \hat{z}^2} = \mathbf{0}, \qquad \epsilon \frac{\partial J_x}{\partial \hat{x}} + \mathrm{i}b\beta J_y + \frac{\partial J_z}{\partial \hat{z}} = 0, \tag{12}$$

and the transition conditions (3)–(6) into

$$E_y - \frac{1}{\sigma} J_y \Big|_{\hat{z}=\pm 1} = 0, \qquad E_x - \frac{1}{\sigma} J_x \Big|_{\hat{z}=\pm 1} = 0, \tag{13}$$

$$-\epsilon \frac{\partial E_z}{\partial \hat{x}} + \frac{\partial E_x}{\partial \hat{z}} + \frac{1}{\sigma} \frac{\partial J_x}{\partial \hat{z}} \Big|_{\hat{z}=\pm 1} = 0, \qquad \mathrm{i}b\beta E_z - \frac{\partial E_y}{\partial \hat{z}} + \frac{1}{\sigma} \frac{\partial J_y}{\partial \hat{z}} \Big|_{\hat{z}=\pm 1} = 0, \tag{14}$$

$$\rho_S = \pm \varepsilon_d E_z|_{\hat{z}=\pm 1}\,. \qquad J_z|_{\hat{z}=\pm 1} = 0. \tag{15}$$

We are interested in $\mathbf{E}$ in the neighbourhood of the strip ($|\hat{x}|, |\hat{z}| > 1$, but $\hat{x}, \hat{z} = O(1)$) and $\mathbf{J}$ in the strip ($-1 < \hat{x}, \hat{z} < 1$). For $-1 < \hat{x} < 1$ with $1-|x| \neq O(\epsilon)$, and for $\hat{z} = O(1)$, the field quantities are only weakly dependent on $\hat{x}$, because $\epsilon \ll 1$. For the numerical values given in Table 1, $\epsilon = O(10^{-3})$. The small parameter ϵ is responsible for boundary layers near the edges $\hat{x} = \pm 1$. From now on we will neglect these effects, meaning that in the field equations (11)–(12), we take $\epsilon = 0$. This implies that all fields are now independent of $\hat{x}$, *i.e.*, $\mathbf{E} = \mathbf{E}(\hat{z})$ and $\mathbf{J} = \mathbf{J}(\hat{z})$. Moreover, since E_x and J_x are governed by an independent set of differential equations and boundary conditions, including $E_x \to 0$ as $|\hat{z}| \to \infty$, these components must be zero. Hence, $J_x = 0$ and $E_x = 0$.

From the problem formulation, it follows that J_y is an even function of $\hat{z}$. Then, $(12)^2$ and the y-component of $(12)^1$, with $\epsilon = 0$, yield

$$J_y(\hat{z}) = \frac{J_y(1)\cosh\left(\hat{z}/\delta\right)}{\cosh\left(1/\delta\right)}, \qquad J_z(\hat{z}) = -\mathrm{i}b\beta\delta \frac{J_y(1)\sinh\left(\hat{z}/\delta\right)}{\cosh\left(1/\delta\right)}, \qquad |\hat{z}| \leq 1. \tag{16}$$

Note that we use the notation $(12)^1$ and $(12)^2$ to signify the first and second equation of (12). We see that J_z satisfies the z-component of $(12)^1$, with $\epsilon = 0$, but not the transition condition $(15)^2$. We will show that this equation is satisfied neglecting vector components of $O(\sqrt{\omega\varepsilon_d/\sigma})$, which is of $O(10^{-3})$ for the numerical values in Table 1. Since the term $\sqrt{\omega\varepsilon_d/\sigma}$ multiplied by $1/\pi\sqrt{2}$ is the ratio of the the wavelength in the dielectric and the skin depth as defined in (29), the wavelength is much larger than the skin depth.

Using $E_y, E_z \to 0$ for $|\hat{z}| \to \infty$, we obtain from $(13)^1$, $(15)^1$, and the y and z-component of $(11)^1$, with $\epsilon = 0$,

$$E_y(\hat{z}) = \frac{1}{\sigma} J_y(1)\, \mathrm{e}^{-\xi(|\hat{z}|-1)}, \qquad E_z(\hat{z}) = \mathrm{sign}(\hat{z})\, \frac{\rho_S}{\varepsilon_{(d)}}\, \mathrm{e}^{-\xi(|\hat{z}|-1)}, \qquad |\hat{z}| \geq 1. \tag{17}$$

Substituting (17) in $(11)^2$, with $\epsilon = 0$, we find

$$\rho_S = \frac{\mathrm{i}b\beta\varepsilon_d}{\sigma\xi}\, J_y(1). \tag{18}$$

Using this relation together with (17) in (14)[2], and requiring $J_y(1) \neq 0$, we arrive at the following dispersion relation for β,

$$-\frac{b^2\beta^2}{\xi} + \xi + \frac{1}{\delta}\tanh(1/\delta) = 0. \tag{19}$$

Recall that ξ depends on β. By the definition of ξ^2, *i.e.*, (10)[2], it follows from (19) that

$$\xi = b^2k_d^2\delta\coth(1/\delta)), \qquad \beta^2 = k_d^2(1 + b^2k_d^2\delta^2\coth^2(1/\delta)). \tag{20}$$

Substitution of (20)[2] in (10)[3] yields an equation for δ,

$$\frac{1}{b^2k_d^2\delta^2} - b^2k_d^2\delta^2\coth^2(1/\delta) = \frac{\sigma\omega\mu_0}{k_d^2}\left(\frac{k_d^2 - k_s^2}{\sigma\omega\mu_0} - \mathrm{i}\right). \tag{21}$$

Having solved this equation, we obtain the field components J_y, J_z, E_y, and E_z, and the surface charge density ρ_S from (16)–(18) and (20). The unknown $J_y(1)$ is incorporated into the prescribed total amplitude I as follows. Neglecting boundary effects near $\hat{x} = \pm 1$, we calculate the total current $\mathcal{I}(y,t)$ passing at time t through the cross section Σ of the strip,

$$\mathcal{I}(y,t) = \int_\Sigma \mathcal{J}_y\,\mathrm{d}S = I\,\mathrm{e}^{\mathrm{i}(\beta y - \omega t)}, \quad I = 4ab\delta\,J_y(1)\,\tanh(1/\delta). \tag{22}$$

Then, the field components J_y, J_z, E_y, and E_z, the x-component H_x of the magnetic field in the dielectric, and the surface charge density ρ_S are given by

$$\rho_S = \frac{\mathrm{i}\beta\varepsilon_d I^{(b)}}{2\sigma\xi\delta}\coth(1/\delta)), \quad H_x(\hat{z}) = \mathrm{sign}(\hat{z})\frac{\mathrm{i}\beta^2 I^{(b)}}{2b^2\sigma\omega\mu_0k_d^2\delta^2}(1-\xi\delta)\,\mathrm{e}^{-\xi(|\hat{z}|-1)}, \tag{23}$$

$$J_y(\hat{z}) = \frac{I^{(b)}}{2b\delta}\frac{\cosh(\hat{z}/\delta)}{\sinh(1/\delta)}, \qquad J_z(\hat{z}) = -\frac{\mathrm{i}\beta I^{(b)}}{2}\frac{\sinh(\hat{z}/\delta)}{\sinh(1/\delta)}, \tag{24}$$

$$E_y(\hat{z}) = \frac{I^{(b)}}{2\sigma b\delta}\coth(1/\delta)\,\mathrm{e}^{-\xi(|\hat{z}|-1)}, \qquad E_z(\hat{z}) = \mathrm{sign}(\hat{z})\frac{\mathrm{i}\beta I^{(b)}}{2\sigma b^2k_d^2\delta^2}\,\mathrm{e}^{-\xi(|\hat{z}|-1)}, \tag{25}$$

where $I^{(b)} = I/2a$ is the total current through a line segment $-b \le z \le b$ of the cross section Σ, the parameters ξ and β are given by (20), and H_x is calculated from Maxwell's equations. Note that the field components H_y and H_z are zero, as are the field components E_x and J_x. Together with the solution δ of (21), Equations (23)–(25) describe the electromagnetic field in and in the neighbourhood of the strip, except near its edges. We show by a dimensional analysis that these expressions amount to the expressions for the electromagnetic field of a perfectly conducting strip.

To approximate δ, we consider (21). From the numerical values in Table 1, it follows that $b^2k_d^2 = O(10^{-4})$ and $(k_d^2 - k_s^2)/\sigma\omega\mu_0 = O(10^{-5})$. Hence, $b^2k_d^2$ is a small parameter in (21), and $(k_d^2 - k_s^2)/\sigma\omega\mu_0$ is much smaller than i. We consider first solutions of (21) for which $\delta^2 = O(1)$. Neglecting terms of $O(b^2k_d^2)$, we obtain from (21) and (20)[2]

$$\delta^2 = \frac{\mathrm{i}}{b^2\sigma\omega\mu_0}, \qquad \beta = k_d. \tag{26}$$

Note that we have chosen $\mathfrak{Re}\,\beta > 0$ in (8). For the numerical values in Table 1, it follows that $\delta^2 = O(10^{-3})$ and $b^2k_d^2\delta^2 = \mathrm{i}\omega\varepsilon_d/\sigma = O(10^{-5})$. Then, it can be seen that (26)[1] satisfies (21) and $\beta = k_d$ up to terms of $O(\omega\varepsilon_d/\sigma)$. The imaginary part of the approximation of β is zero, and hence the wave is considered as purely propagating. Since we disregard reflections and boundary effects at the ends of the stripline, we may consider it as infinitely long. We note that the imaginary part of β/k_d is of $O(\omega\varepsilon_d/\sigma)$, and hence, the wave is attenuated at a distance of the order of 10^5 times λ_d. In practical cases this distance is much larger than the length of the stripline. Since the real part of the approximation of β equals k_d, the wavelength $\lambda = \lambda_d$ of the propagating wave is much larger than the width $2a$. This corroborates our assumption that the wave propagates in y-direction only.

Secondly, we consider solutions of (21) for which $b^2k_d^2\delta^2 = O(1)$. Neglecting terms of $O(b^2k_d^2)$, we obtain from (21) and (20)[2]

$$\delta^4 = \frac{\mathrm{i}\sigma\omega\mu_0}{b^2k_d^4}, \qquad \beta = (1+\mathrm{i})\sqrt{\frac{\sigma\omega\mu_0}{2}}\,. \tag{27}$$

For the numerical values in Table 1, $|b^2k_d^2\delta|^2 \geq 13{\cdot}6$. Moreover, since $b\,\mathrm{Im}\,\beta \geq 9{\cdot}64$, the wave is attenuated at distances of the order of b. Hence, the wave will not propagate. Therefore, we exclude this solution and consider only the solution (26).

Neglecting terms of $O(\omega\varepsilon_d/\sigma)$ in ρ_S, H_x, and E_z as in δ and β, we obtain from (23), (24)[1] and (25)[2] the expressions (32) and (33)[1]. From (24) and (25), it can be shown that J_z and E_y are of $O(\sqrt{\omega\varepsilon_d/\sigma})$ $(= O(10^{-3})$, see Table 1) with respect to J_y and E_z, respectively, for $\hat{z} = O(1)$. Therefore, we put $J_z = 0$ and $E_y = 0$. Then, J_z satisfies the transition condition (15)[2].

Now, let us consider the field component J_y in (24). In (10)[3], we have chosen $-\pi/2 < \arg\delta \leq \pi/2$, and hence, $1/\delta = A(1-\mathrm{i})$, where $A = b\sqrt{2\sigma\omega\mu_0}$. The numerical values in Table 1 yield $A = 19{\cdot}3$. For $\hat{z} = O(1)$, but $\hat{z} \neq O(1/A)$, we find

$$J_y(\hat{z}) = \frac{I^{(b)}A(1-\mathrm{i})}{2b}\mathrm{e}^{-A(1-|\hat{z}|)(1-\mathrm{i})}, \tag{28}$$

neglecting terms of $O(\mathrm{e}^{-2A}) = O(10^{-17})$. For $\hat{z} = O(1/A)$, we find $J_y(\hat{z}) = 0$, neglecting terms of $O(\mathrm{e}^{-A})$. We can then use the expression (28) for $\hat{z} = O(1)$. Since $A \gg 1$, it follows from (28) that J_y decays very fast with $\hat{z}$, such that it is in fact restricted to very thin layers near $\hat{z} = \pm 1$. We define the thickness $\hat{\delta}_{\mathrm{skin}}$ of each of these layers, relative to half the thickness $2b$ of the stripline, as the value of $1 - |\hat{z}|$ for which real part of the power of the exponent in (28) equals -1. Then, $\hat{\delta}_{\mathrm{skin}} = 1/A$. The dimensional skin depth δ_{skin} of the two layers at $\hat{z} = \pm 1$ together is given by

$$\delta_{\mathrm{skin}} = 2b\,\hat{\delta}_{\mathrm{skin}} = \sqrt{\frac{2}{\sigma\omega\mu_0}}\,. \tag{29}$$

For the numerical values in Table 1, the skin depth is at most 5% of the thickness $2b$ of the stripline ($\hat{\delta}_{\mathrm{skin}} \leq 0{\cdot}052$).

We have obtained (28) by neglecting terms of $O(e^{-A})$. For the numerical values in Table 1, $\mathrm{e}^{-A} = O(10^{-9})$, which is $O(\omega\varepsilon_d/\sigma)$. In the expressions for ρ_S, H_x, E_z, and J_y, we have neglected terms of order $O(\sqrt{\omega\varepsilon_d/\sigma})$. However, ρ_S, H_x, and E_z are no longer dependent on σ and ω, whereas J_y is still dependent on these parameters. Hence, in the expressions of ρ_S, H_x, and E_z, we have taken the limit $\sqrt{\omega\varepsilon_d/\sigma} \to 0$, while in J_y, we have not taken the limit

$O(e^{-A}) \to 0$ or $A \to \infty$. The latter limit can only be taken in a distributional sense. We find

$$\lim_{A\to\infty} J_y(\hat{z}) = \Lambda\left(\delta_f(\hat{z}-1) + \delta_f(\hat{z}+1)\right), \tag{30}$$

where δ_f is the delta function and Λ is a constant. This means that

$$\lim_{A\to\infty} \int_{-\infty}^{\infty} J_y(\hat{z})\varphi(\hat{z})\,\mathrm{d}\hat{z} = \Lambda\left(\varphi(1) + \varphi(-1)\right), \tag{31}$$

for every function $\varphi \in C_c^\infty(\mathbb{R})$. Here, $\varphi \in C_c^\infty(\mathbb{R})$ is the space of all infinitely many times continuously differentiable functions on $\mathbb{R}$ with compact support. Taking φ such that $\varphi(\hat{z}) = 1$ for $-1 \le \hat{z} \le 1$, we obtain $\Lambda = I^{(b)}/2b$.

Summarizing, we have deduced the following asymptotic expressions for the field components and the surface charge density, which are valid for $\hat{z}, \hat{x} = O(1)$ and $1 - |\hat{x}| \neq O(\varepsilon)$:

$$\rho_S = \frac{\sqrt{\varepsilon_d\mu_0}\, I^{(b)}}{2}, \qquad H_x(\hat{z}) = \operatorname{sign}(\hat{z})\frac{I^{(b)}}{2}, \tag{32}$$

$$E_z(\hat{z}) = \operatorname{sign}(\hat{z})\sqrt{\frac{\mu_0}{\varepsilon_d}}\,\frac{I^{(b)}}{2}, \qquad J_y(\hat{z}) = \frac{I^{(b)}}{2b}\left(\delta_f(\hat{z}-1) + \delta_f(\hat{z}+1)\right), \tag{33}$$

$$\beta = k_d\,. \tag{34}$$

Note that the other field components are zero. These expressions can be interpreted as follows. Taking the limits $\sqrt{\omega\varepsilon_d/\sigma} \to 0$ and $A = b\sqrt{2\sigma\omega\mu_0} \to \infty$ corresponds to taking the limit $\sigma \to \infty$, *i.e.*, the stripline is a perfect conductor. Moreover, since the thickness $2b$ is much smaller than all other length scales and the current is located at the boundary, the stripline can be modeled as infinitely thin. The boundary conditions of an infinitely thin perfectly conducting stripline at its boundaries $z = \pm 0$ follow from (32) and (33) with $\hat{z} = \pm 1$. We see that the tangential component of the electric field and the normal component of the magnetic field vanish at the surface of the stripline, which corresponds with the vanishing of the fields inside a perfect conductor and the continuity of the fields over the boundaries $\hat{z} = \pm 1$.

3. Modeling a stripline in a stripline environment

In this section we consider a long thin conducting stripline in a stripline environment, as described in Section 1, and we assume that the current in the stripline is a propagating wave in the y-direction. Since the distance $2h$ between the ground plates and the sizes of these plates are much larger than the thickness of the stripline, we can model the stripline as infinitely long, infinitely thin, and perfectly conducting, based on the results of the previous section. However, we now take $J_y = J_y(x)$ instead of $I^{(b)}$ as in the previous section, see (32) and (33), because we wish to account also for the edge effects at $x = \pm a$.

Because of symmetry with respect to the plane $z = 0$, we take $\mathbf{x} \in G^+ = \{\mathbf{x} \in \mathbb{R}^3 \mid x, y \in \mathbb{R}, 0 < z < h\}$, see Figure 1. The current on the stripline is described by

$$\mathcal{J}(\mathbf{x}, t) = \mathbf{J}(x)e^{i(\beta y - \omega t)} = J_y(x)e^{i(\beta y - \omega t)}\mathbf{e}_y, \tag{35}$$

for $\mathbf{x} \in S = \{\mathbf{x} \in \mathbb{R}^3 \mid -a < x < a, y \in \mathbb{R}, z = 0\}$. Here, $\mathfrak{Re}\,\beta > 0$. Although we have shown in the previous section that $\beta = k$, where $k = \omega\sqrt{\varepsilon\mu_0}$ is the wave number in the dielectric, we put β as the (unknown) wave number and we will show that $\beta = k$ for transverse electromagnetic waves.

In the literature, boundary-value problems for the electric potential are deduced for several types of stripline, see [1, Chapter 3]. Therefore, we only summarize the steps to arrive at the boundary-value problem for this potential. The electric field $\boldsymbol{\mathcal{E}}$ and the magnetic field $\boldsymbol{\mathcal{H}}$ in the dielectric are described by Maxwell's equations. They are also planar waves in the y-direction, as is the charge density ϱ_S on the stripline; see (9). Then, **E** and **H** are described by the source-free Maxwell equations for planar waves of the type (35). These equations are supplemented by boundary conditions. The ground plates are assumed to be perfectly conducting, which implies that the tangential component of the electric field and the normal component of the magnetic field vanish at the ground plates. At the stripline, the boundary conditions from the results are obtained from (32) and (33), where $z = \pm 1$ is replaced by $z = 0^{\pm}$ and $I^{(b)}$ is replaced by $J_y(x)$. At the symmetry line $|x| > a, z = 0^+$, the boundary conditions follow from the behaviour of **E** and **H**, *i.e.*, H_x, H_y, and E_z are odd in z, and E_x, E_y, and H_z are even in z.

We assume that the planar wave in the stripline environment is a transverse electromagnetic (TEM) wave, which implies $E_y = 0$ and $H_y = 0$. Then, the x-components and the z-components of Maxwell's equations (the law of Faraday and the law of Ampère-Maxwell) are satisfied if and only if $\beta^2 = \omega^2 \varepsilon \mu_0 = k^2$. Hence, the stripline can only support a TEM wave if $\beta^2 = k^2$. This is in correspondence with $(26)^2$ for the asymptotic solution for a stripline of finite thickness and finite conductivity. From the y-component of Maxwell's equations, it follows that **E** and **H** are irrotational. Then, **E** and **H** are both conservative, because G^+ is simply connected. This implies that **E** and **H** can be written as $\mathbf{E} = -\operatorname{grad}\phi$ and $\mathbf{H} = -\operatorname{grad}\psi$, where the scalar functions ϕ and ψ are called the electric and magnetic potential, respectively. From $\operatorname{div}\boldsymbol{\mathcal{E}} = 0$ and $\operatorname{div}\boldsymbol{\mathcal{H}} = 0$, it follows that they are harmonic,

$$\Delta\phi = 0, \quad \Delta\psi = 0, \tag{36}$$

and from the y-component of Maxwell's equations, it follows that they satisfy the Cauchy-Riemann relations

$$-\frac{\partial\psi}{\partial z} = \sqrt{\frac{\varepsilon}{\mu_0}}\,\frac{\partial\phi}{\partial x}, \quad \frac{\partial\psi}{\partial x} = \sqrt{\frac{\varepsilon}{\mu_0}}\,\frac{\partial\phi}{\partial z}. \tag{37}$$

Because of (37), we do not need to solve the boundary-value problems for both ϕ and ψ. Therefore, we deduce and solve the boundary-value problem for ϕ only. The boundary conditions for ϕ are deduced from the aforementioned boundary conditions for the electric and magnetic field. Then, the boundary-value problem for ϕ is given by

$$\begin{aligned}
&\Delta\phi = 0, && |x| < \infty,\ 0 < z < h,\\
&\frac{\partial\phi}{\partial x} = 0, && z = h,\\
&\frac{\partial\phi}{\partial z} = 0, && z = 0^+,\ |x| > a,\\
&\frac{\partial\phi}{\partial z} = -\sqrt{\frac{\mu_0}{\varepsilon}}\,\frac{J_y(x)}{2}, && z = 0^+,\ |x| < a,\\
&\frac{\partial\phi}{\partial x} = 0, && z = 0^+,\ |x| < a.
\end{aligned} \tag{38}$$

The solution to this problem is not unique. Since the tangential derivatives of ϕ at $z = h$ and at $z = 0,\ |x| < a$ are zero, we may prescribe a constant potential at these boundaries. Without loss of generality, we put $\phi = 0$ at $z = h$ and $\phi = \phi_0$ at $z = 0,\ |x| < a$, thus arriving at a mixed (Dirichlet-Neumann) boundary-value problem for ϕ. The potential ϕ_0 is related to the total current amplitude I by $(38)^4$ and hence, it can be interpreted as the generator of the current on the stripline. To normalize the problem for ϕ, we introduce

$$x = a\hat{x}, \quad z = h\hat{z}, \quad \phi = \phi_0\hat{\phi}, \quad J_y = \frac{2\phi_0}{h}\sqrt{\frac{\mu_0}{\varepsilon}}\ \hat{J}, \tag{39}$$

where $\hat{J}$ is a function of $\hat{x}$. Then, the boundary-value problem for $\hat{\phi}$ is given by

$$\begin{aligned}
&\frac{\partial^2\hat{\phi}}{\partial\hat{x}^2} + \frac{1}{\zeta^2}\frac{\partial^2\hat{\phi}}{\partial\hat{z}^2} = 0, && |\hat{x}| < \infty, 0 < \hat{z} < 1,\\
&\hat{\phi} = 0, && \hat{z} = 1,\\
&\frac{\partial\hat{\phi}}{\partial\hat{z}} = 0, && \hat{z} = 0^+,\ |\hat{x}| > 1,\\
&\frac{\partial\hat{\phi}}{\partial\hat{z}} = -\hat{J}(\hat{x}), && \hat{z} = 0^+,\ |\hat{x}| < 1,\\
&\hat{\phi} = 1, && \hat{z} = 0^+,\ |\hat{x}| < 1,
\end{aligned} \tag{40}$$

where $\zeta = h/a$. In the analysis that follows we omit the hats. From $(40)^1$ we conclude that ϕ depends on ζ, and therefore J also depends on ζ. To show these dependencies explicitly, we write $\phi(x, z; \zeta)$ and $J(x; \zeta)$.

4. Calculation of potential and current

To solve the boundary-value problem (40) for ϕ, we apply the Fourier transformation with respect to x, *i.e.*,

$$\mathcal{F}\{f(x,\cdot)\ ;\ x \to s\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,\cdot)\,\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}x. \tag{41}$$

to the differential equation for ϕ and the boundary condition at $z = 1$. Then it follows that

$$\begin{aligned}
\phi(x, z; \zeta) &= \mathcal{F}^{-1}\{C(s;\zeta)\sinh(\zeta s(1-z))\ ;\ s \to x\} =\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} C(s;\zeta)\sinh(\zeta s(1-z))\,\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}s, \quad 0 < z < 1,\ |x| < \infty,
\end{aligned} \tag{42}$$

where C is an unknown function and $\mathcal{F}^{-1}$ is the inverse Fourier transformation. We distinguish two approaches to calculate C and J. In the first approach, we choose the conditions $(40)^{3,4}$ to calculate the potential ϕ, *i.e.*, to express ϕ in terms of the current. The current is calculated from the condition for ϕ on the plate, *i.e.*, $(40)^5$. In the second approach we choose the conditions $(40)^{3,5}$ to calculate the potential ϕ. The current is calculated from the condition $(40)^4$.

4.1. First approach

The boundary conditions (40)[3,4] yield an integral equation for $C(s;\zeta)$,

$$\mathcal{F}^{-1}\{\zeta s C(s;\zeta)\cosh(\zeta s)\ ;\ s\to x\} = \begin{cases} J(x;\zeta), & |x|<1, \\ 0, & |x|>1. \end{cases} \tag{43}$$

Applying the Fourier transformation to both sides of the equation, we obtain

$$C(s;\zeta) = \frac{1}{\sqrt{2\pi}\,\zeta s\cosh(\zeta s)}\int_{-1}^{1} J(u;\zeta)\,\mathrm{e}^{\mathrm{i}su}\,\mathrm{d}u. \tag{44}$$

Substituting this expression in (42) and reversing the order of integration, we obtain

$$\phi(x,z;\zeta) = \int_{-1}^{1} J(u;\zeta)k(x-u,z;\zeta)\,\mathrm{d}u, \tag{45}$$

where

$$k(\alpha,z;\zeta) = \mathcal{F}_{\cos}^{-1}\left\{\frac{\sinh(\zeta s(1-z))}{\sqrt{2\pi}\,\zeta s\cosh(\zeta s)}\ ;\ s\to\alpha\right\} = \\ = \frac{1}{2\pi\zeta}\log\left[\frac{\cosh\left(\frac{\alpha\pi}{2\zeta}\right)+\sin\left(\frac{(1-z)\pi}{2}\right)}{\cosh\left(\frac{\alpha\pi}{2\zeta}\right)-\sin\left(\frac{(1-z)\pi}{2}\right)}\right], \tag{46}$$

see Erdélyi *et al.* [10, Table 1.9 (34), p. 33]. Note that $\mathcal{F}_{\cos}$ is the inverse Fourier cosine transformation defined by

$$\mathcal{F}_{\cos}\{f(x,\cdot)\ ;\ x\to s\} = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} f(x,\cdot)\,\cos(sx)\,\mathrm{d}x. \tag{47}$$

Substituting (45) in (40)[5], we obtain a Fredholm equation of the first kind for J,

$$\mathcal{K}_\zeta J = 1, \qquad \text{on } [-1,1], \tag{48}$$

where $\mathcal{K}_\zeta$ is defined by

$$(\mathcal{K}_\zeta J)(x) = \int_{-1}^{1} J(u;\zeta)\,k_0(x-u;\zeta)\mathrm{d}u, \qquad |x|<1, \tag{49}$$

and

$$k_0(\alpha;\zeta) = k(\alpha,0;\zeta) = \frac{1}{\pi\zeta}\log\left|\coth\frac{\alpha\pi}{4\zeta}\right|. \tag{50}$$

Let for any function f the function $f^\vee$ be defined by $f^\vee(x) = f(-x)$. Then, it can be shown that $\mathcal{K}_\zeta J^\vee = (\mathcal{K}_\zeta J)^\vee = 1$ on $(-1,1)$. Hence, $J(x;\zeta)$ is even in x, if the solution of (48) is unique.

We do not consider the question of the existence and uniqueness of the solution of (48). We note only that the solution for J in the boundary-value problem (40) is not an element of

$L_2[-1,1]$, because J has square-root singularities in ± 1. This can be shown by solving the Laplace equation for ϕ near $x = 1$ in local cylindrical coordinates. Then, the condition that the electric energy should be finite yields the behaviour of ϕ, and therewith J, near $x = 1$.

4.2. Second approach

The boundary conditions (40)[3,5] yield a system of two integral equations for $C(s;\zeta)$,

$$\begin{aligned} &\mathcal{F}^{-1}\{C(s;\zeta)\sinh(\zeta s)\ ;\ s\to x\} = 1, \qquad &|x|<1,\\ &\mathcal{F}^{-1}\{s\,C(s;\zeta)\cosh(\zeta s)\ ;\ s\to x\} = 0, \qquad &|x|>1. \end{aligned} \tag{51}$$

This system is called a system of dual integral equations, see Sneddon [11, Chapter 4]. It can be seen that if $C(s;\zeta)$ is a solution to this system, $-C(-s;\zeta)$ is also. Hence, $C(s;\zeta)$ is odd in s, if the solution is unique. Then, the Fourier transforms can be replaced by Fourier cosine transforms. To solve the system, we write

$$s\,C(s;\zeta)\cosh(\zeta s) = \int_{t=-1}^{1} g(t;\zeta)B(t,s)\,\mathrm{d}t, \tag{52}$$

where $B(t,s)$ is a given function, which is even in s and $g(t;\zeta)$ is an unknown function. Substituting this expression in (51)[2] and reversing the order of integration, we obtain

$$\int_{t=-1}^{1} g(t;\zeta)\mathcal{F}_{\cos}^{-1}\{B(t,s)\ ;\ s\to x\}\,\mathrm{d}t = 0, \qquad |x|>1. \tag{53}$$

This equation is satisfied, if we assume that

$$\mathcal{F}_{\cos}^{-1}\{B(t,s)\ ;\ s\to x\} = \frac{H(|t|-|x|)}{D(t,x)}, \tag{54}$$

where D depends on the choice of B or *vice versa*. Since $B(t,s)$ is even in s, $D(t,x)$ has to be even in x. Substituting (52) in the expression for ϕ, *i.e.*, (42), we obtain

$$\phi(x,z;\zeta) = \int_{t=-1}^{1} g(t;\zeta)G(x,z,t;\zeta)\,\mathrm{d}t, \tag{55}$$

where $G(x,z,t;\zeta)$ is defined by

$$G(x,z,t;\zeta) = \mathcal{F}_{\cos}^{-1}\left\{\frac{\sinh(\zeta s(1-z))}{s\cosh(\zeta s)}B(t,s)\ ;\ s\to x\right\}. \tag{56}$$

By (54), G can be written as

$$G(x,z,t;\zeta) = \zeta\int_{u=-|t|}^{|t|}\frac{1}{D(t,u)}\,k(x-u,z;\zeta)\,\mathrm{d}u. \tag{57}$$

Then, (51)[1], *i.e.*, $\phi = 1$ for $|x|<1$, turns into a Fredholm equation of the first kind for g,

$$\int_{t=-1}^{1} g(t;\zeta)G(x,0,t;\zeta)\mathrm{d}t = 1, \qquad |x|<1. \tag{58}$$

After calculation of g from this equation, ϕ is known and J can be calculated from (40)[4]. Since $C(s; \zeta$ is even in s, $\phi(x, z; \zeta)$ is even in x and therefore, $J(x; \zeta)$ is even in x.

To choose $D(t, x)$, we note first that $D(t, x)$ should be even in x and the transforms in (54) and (56) should exist. Examples of choices of $D(t, x)$ are $D(t, x) = 1$ with $B(t, s) = \sqrt{\frac{2}{\pi}} \frac{\sin(s|t|)}{s}$ and $D(t, x) = \sqrt{\pi(t^2 - x^2)/2}$ with $B(t, s) = \mathcal{J}_0(ts)$, *i.e.*, the Bessel function of the first kind of index 0. For the first choice, we obtain

$$G(x, z, t; \zeta) = \zeta \; [K(|t| + x, z; \zeta) + K(|t| - x, z; \zeta)] , \tag{59}$$

where K is defined by

$$K(\alpha, z; \zeta) = \frac{1}{\pi\zeta} \int_{s=0}^{\infty} \frac{\sin(\alpha s) \sinh(\zeta s(1-z))}{s^2 \cosh(\zeta s)} \mathrm{d}s. \tag{60}$$

Note that $\partial K/\partial\alpha = k$, where the function k is given by (46). The function K can be written as a series expansion by carrying out a contour integration in the complex plane,

$$\operatorname{sign}(\alpha)\, K(\alpha, z; \zeta) = \frac{1}{2}(1-z) - \frac{1}{\pi^2} \sum_{n=0}^{\infty} \frac{\cos((n+\frac{1}{2})\pi z)}{(n+\frac{1}{2})^2} \exp\left(-\frac{|\alpha|}{\zeta}\left(n+\frac{1}{2}\right)\pi\right). \tag{61}$$

4.3. A RELATION BETWEEN J AND g

In the first approach, we solve an integral equation for J, while in the second approach, we solve an integral equation for g. A relation between J and g can be deduced as follows. We substitute the expression (52) for C of the second approach in the integral equation of the first approach, *i.e.*, (43). Reversing the order of integration and using (54), we obtain

$$J(x; \zeta) = \zeta \left(\int_{t=|x|}^{1} + \int_{t=-1}^{-|x|} \right) \frac{g(t; \zeta)}{D(t, x)} \,\mathrm{d}t, \quad |x| < 1. \tag{62}$$

By this relation, it can be shown that the two approaches yield the same result for ϕ and J. Substituting (57) in (58) and using (62), we obtain the integral equation for J as obtained in the first approach, see (48).

5. Asymptotic solutions

In applications of stripline technology, the ratio $\zeta = h/a$ is usually chosen of order 1. Here we consider the asymptotic cases $\zeta \ll 1$ and $\zeta \gg 1$. In Section 7 we show that these asymptotic cases yield reasonable approximations for the impedance $\zeta \lesssim 0{\cdot}2$ and $\zeta \gtrsim 3{\cdot}2$.

We consider the case $\zeta = h/a \ll 1$ first. We approximate $G(x, 0, t; \zeta)$ in (58) by

$$G(x, 0, t; \zeta) = \zeta \; \mathcal{F}_{\cos}^{-1} \{B(t, s) \;;\; s \to x\} = \zeta \, \frac{H(|t| - |x|)}{D(t, x)} \;. \tag{63}$$

Then, by (62), (58) turns into

$$J(x; \zeta) = 1, \qquad |x| < 1, \tag{64}$$

Hence, the current on the strip is (asymptotically) uniform. Then, using the expression (45) for ϕ in the first approach, we obtain

$$\phi(x, z; \zeta) = K(1 + x, z; \zeta) + K(1 - x, z; \zeta), \tag{65}$$

where K is defined by (60) with $\partial K/\partial\alpha = k$. It can be shown that ϕ satisfies the differential equation and the boundary conditions for ϕ in (40), except $(40)^5$. The latter serves as a measure for the quality of the approximation. Numerically, it can be shown that for $\zeta = 0{\cdot}05$, $|\phi(x, 0^+; \zeta) - 1| \leq 0{\cdot}01$ for $0 \leq x \leq 0{\cdot}8$ and $|\phi(x, 0^+; \zeta) - 1| \leq 0{\cdot}05$ for $0 \leq x \leq 0{\cdot}89$. The impedance of the stripline is given by

$$Z = \frac{\zeta}{4\sqrt{\varepsilon_r}}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left(\int_{x=0}^{1} J(x; \zeta)\mathrm{d}x\right)^{-1} = \frac{30\pi}{\sqrt{\varepsilon_r}}\zeta, \tag{66}$$

where $\varepsilon = \varepsilon_r\varepsilon_0$ and $\sqrt{\mu_0/\varepsilon_0} = 120\pi$, see Table 1. For $\zeta \ll 1$, This result is approximately equal to the approximations of Pozar [12, p. 156] and Howe [13, p. 34–35] for $\zeta < 1/0{\cdot}35$.

Next, we consider the case $\zeta = h/a \gg 1$. We approximate k_0 in (50) by

$$k_0(\alpha; \zeta) = \frac{1}{\pi\zeta}\log\frac{4\zeta}{|\alpha|\pi}\,. \tag{67}$$

Since J has square-root singularities in ± 1, we define

$$\tilde{J}(x; \zeta) = \sqrt{1 - x^2}\,J(x; \zeta), \qquad |x| < 1. \tag{68}$$

Then, with $x = \cos\theta$ and $u = \cos\theta'$, (48) turns into

$$\frac{1}{\pi\zeta}\int_{\theta'=0}^{\pi}\tilde{J}(\cos\theta'; \zeta)\log\frac{4\zeta}{|\cos\theta - \cos\theta'|\pi}\,\mathrm{d}\theta' = 1, \qquad 0 < \theta < \pi. \tag{69}$$

Note that $\tilde{J}$ is even as function of both θ and x. Expanding $\tilde{J}(\cos\theta; \zeta)$ into a Fourier series,

$$\tilde{J}(\cos\theta; \zeta) = \sum_{n=0}^{\infty}\alpha_n(\zeta)\,\cos 2n\theta, \tag{70}$$

and writing the logarithmic kernel as

$$\log|\cos\theta - \cos\theta'| = -\log 2 - 2\sum_{l=1}^{\infty}\frac{1}{l}\cos l\theta\cos l\theta', \tag{71}$$

we obtain

$$\frac{\alpha_0(\zeta)}{\zeta}\log\frac{8\zeta}{\pi} + \frac{1}{2\zeta}\sum_{n=1}^{\infty}\frac{\alpha_n(\zeta)}{n}\,\cos 2n\theta = 1, \qquad 0 \leq \theta \leq \pi. \tag{72}$$

Identifying corresponding Fourier coefficients, we arrive at

$$\alpha_0(\zeta) = \frac{\zeta}{\log\frac{8\zeta}{\pi}}, \qquad \alpha_n(\zeta) = 0,\ n \geq 1. \tag{73}$$

Then, the current J and the impedance Z are given by

$$J(x;\zeta) = \frac{\zeta}{\log\left(\frac{8\zeta}{\pi}\right)\sqrt{1-x^2}}\ , \qquad Z = \frac{60}{\sqrt{\varepsilon_r}}\log\frac{8\zeta}{\pi}\,. \tag{74}$$

We see that the current equals a constant times its edge behaviour, *i.e.*, a square-root singularity. Furthermore, the impedance equals the approximation of Howe [13, p. 34–35] for $\zeta > 1/0{\cdot}35$ and is approximately equal to the approximation of Pozar [12, p. 156].

6. Numerical solution

For a numerical solution, we consider the integral equation (48) for J of the first approach. Using that J and $\mathcal{K}_\zeta J$ are even, and substituting (68), $x = \cos\theta$, and $u = \cos\theta'$ in (48), we obtain

$$\int_{\theta'=0}^{\pi/2} \tilde{J}(\cos\theta';\zeta)\left[k_0(\cos\theta-\cos\theta';\zeta)+k_0(\cos\theta+\cos\theta';\zeta)\right]\mathrm{d}\theta' = 1,\ \ 0\le\theta\le\pi/2. \tag{75}$$

To solve this equation numerically, we use the method of collocation. First, we expand $\tilde{J}$ into a Fourier series as in (70). Then, we choose a finite number of points $0 < \theta_m < \pi/2$, $m = 0, 1, ..., N$, and we require

$$\sum_{n=1}^{N}\alpha_n(\zeta)\int_{\theta'=0}^{\pi/2}\cos((2n-2)\theta')\left[k_0(\cos\theta_m-\cos\theta';\zeta)+k_0(\cos\theta_m+\cos\theta';\zeta)\right]\mathrm{d}\theta' = 1,$$
$$m = 0, 1, ..., N. \tag{76}$$

In matrix notation, this can be written as $\mathcal{Z}\alpha = \mathcal{V}$, where $\alpha = (\alpha_0, ..., \alpha_N)^T$, $\mathcal{V} = (1, ..., 1)^T$ and

$$\mathcal{Z}(m,n) = \int_{\theta'=0}^{\pi/2}\cos((2n-2)\theta')\left[k_0(\cos\theta_m-\cos\theta';\zeta)+k_0(\cos\theta_m+\cos\theta';\zeta)\right]\mathrm{d}\theta'. \tag{77}$$

The integrals are computed by a Newton-Coates integration rule applied to subdivisions of the interval $(0,\pi/2)$. Near the singularity in $\theta' = \theta_m$, more subdivisions are used. Using (68) and (70), we write J as

$$J(x;\zeta) = \frac{1}{\sqrt{1-x^2}}\sum_{n=0}^{N}\alpha_n(\zeta)T_{2n}(x), \qquad |x| < 1. \tag{78}$$

The impedance is computed as in (66), which yields $Z = 60\zeta/\alpha_1(\zeta)\sqrt{\varepsilon_r}$.

7. Numerical results

From Table 2 and Figure 2, we see that three collocation points yield a reasonable approximation of the current and the impedance for $\zeta = 0{\cdot}05$. Sets of 4 and 7 collocation points yield comparable results. For 9 collocation points, the algorithm is not stable anymore. The function $J(x; 0{\cdot}05)$ is almost equal to 1 on the larger part of the interval $[0, 1]$, as in the

Table 2. Values of the expansion coefficients α_n and the impedances for four collocation point sets $x_m = \cos\theta_m$ for $\zeta = 0{\cdot}05$. Set 1: 0·1, 0·5, 0·9; Set 2: 0·2, 0·4, 0·6, 0·8; Set 3: 0·1, 0·2, 0·35, 0·5, 0·65, 0·8, 0·9; Set 4: 0·1, 0·2, 0·3, 0·4, 0·5, 0·6, 0·7, 0·8, 0·9.

Set	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	$Z\sqrt{\varepsilon_r}$(ohm)
1	0·656	−0·381	−0·038							4·575
2	0·660	−0·379	−0·046	−0·008						4·544
3	0·659	−0·380	−0·044	−0·001	0·008	0·006	0·002			4·553
4	0·856	0·014	0·341	0·351	0·296	0·208	0·116	0·046	0·010	3·503

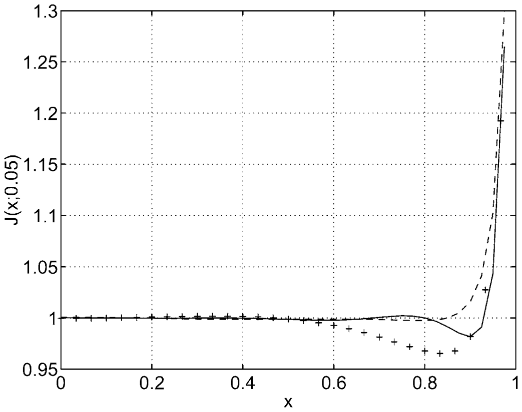

Figure 2. The current distribution on right half of the strip ($0 \le x < 1$) for $\zeta = 0.05$ for three sets of collocation points. Solid curve: collocation points 0·1, 0·2, 0·35, 0·5, 0·65, 0·8, 0·9; Dashed curve: collocation points 0·2, 0·4, 0·6, 0·8; Crosses: collocation points 0·1, 0·5, 0·9.

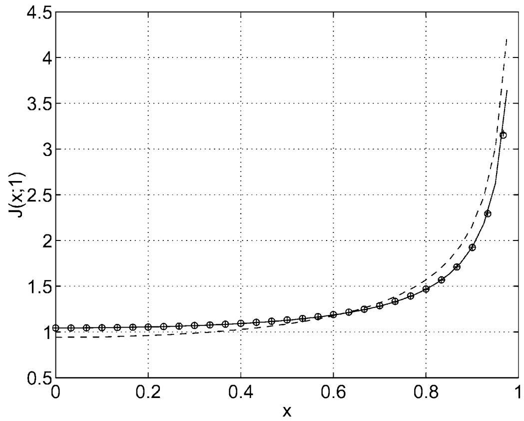

Figure 3. The current distribution on the strip ($0 < x < 1$) for $\zeta = 1$. Dashed line: collocation point 0·5; Circles: collocation points 0·3, 0·7; Crosses: collocation points 0·1, 0·5, 0·9; Solid line: collocation points 0·2, 0·4, 0·6, 0·8.

asymptotic solution (64). Furthermore, the impedance differs only 3·4% from the asymptotic approximation (66).

From Table 3 and Figure 3, we see that two collocation points yield a good approximation of the current and the impedance for $\zeta = 1$. Furthermore, it can be seen that the current equals a constant times its edge behaviour plus a small correction term.

We do not show a result for $\zeta \gg 1$. For $\zeta = 20$, only one collocation point is needed ($x_m = \cos\theta_m = 0{\cdot}5$), and the numerical and asymptotic solution match perfectly.

Figure 4 shows the numerical solution and the two asymptotic solutions. The asymptotic solution for $\zeta \ll 1$ matches with the numerical solution for $\zeta \lesssim 10^{-0{\cdot}7} \approx 0{\cdot}2$, while the asymptotic solution for $\zeta \gg 1$ matches with the numerical solution for $\zeta \gtrsim 10^{0{\cdot}5} \approx 3{\cdot}2$.

Figure 5 shows the numerical solution and the formulae of Pozar [12, p. 156] and Howe [13, p. 34–35]. We see that the curves match perfectly, except for larger values of ζ ($\zeta > 15$), where the formula of Pozar differs from both the numerical solution and the formula of Howe (about 4% at $\zeta = 30$).

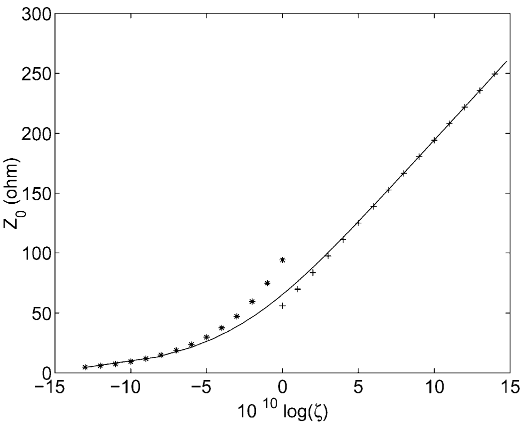

Figure 4. The impedance $Z_0 = Z\sqrt{\varepsilon_r}$ as a function of $10\,{}^{10}\log\zeta$. Solid line: Numerical approximation; Stars: asymptotic approximation (66) for $\zeta \ll 1$; Crosses: asymptotic approximation (74) for $\zeta \gg 1$.

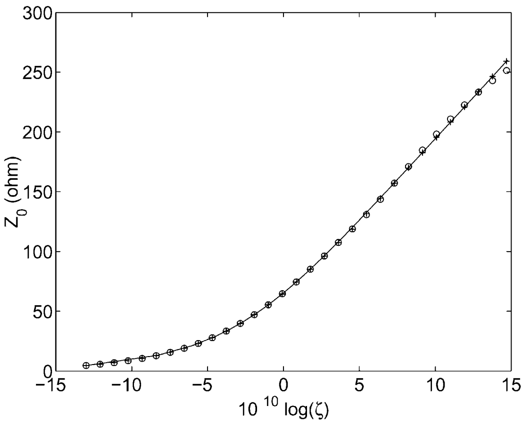

Figure 5. The impedance $Z_0 = Z\sqrt{\varepsilon_r}$ as a function of the $10\,{}^{10}\log\zeta$. Solid line: numerical approximation, Crosses: impedance formula of Howe, Circles: impedance formula of Pozar.

Table 3 Values of the expansion coefficients α_n and the impedances for four collocation point sets x_m ($\zeta = 1$). Set 1: 0·5; Set 2: 0·3, 0·7; Set 3: 0·1, 0·5, 0·9; Set 4: 0·2, 0·4, 0·6, 0·8.

Set	α_1	α_2	α_3	α_4	$Z\sqrt{\varepsilon_r}$(ohm)
1	0·942				63·72
2	0·917	−0·127			65·46
3	0·918	−0·123	0·004		65·40
4	0·918	−0·123	0·004	-2×10^{-5}	65·40

8. Results and conlusions

Asymptotic expressions have been deduced for the electromagnetic fields in and near a long thin conducting stripline, embedded in a dielectric layer, where the current is assumed to be a propagating wave in the length direction, with prescribed total amplitude. Reflections and boundary effects in the end sections are disregarded. Neglecting edge effects at $x = \pm a$ on basis of $b/a = O(10^{-3})$, and consequently, neglecting terms of $O(\sqrt{\omega\epsilon_d/\sigma}) = O(10^{-3})$, we find that, in the neighbourhood of the stripline, the electric and magnetic field in the dielectric only have components in the z- and x-direction, respectively. Moreover, these components and the surface-charge density at $z = \pm b$ depend only on the prescribed total amplitude of the current, the permittivity of the dielectric layer and the permeability of vacuum. The current in the stripline has only a component in the y-direction, and its wave number equals the wave number in the dielectric layer. Hence, the wavelength of the current is much larger than the width $2a$ of the stripline, which corroborates our assumption that the current propagates in the y-direction only. Moreover, since the wave number of the current is real, the fields are not attenuated in the y-direction. Then, we may consider the stripline as infinitely long, because we disregard reflections and boundary effects at the ends of the stripline. We have shown that the current decays exponentially, with exponent $-A\,z/b$ and $A = b\sqrt{2\sigma\omega\mu_0} \approx 20$, from the boundaries $z \pm b$, such that this current is restricted to very thin layers near these boundaries. The layers have total characteristic thickness $\delta_{\text{skin}} = \sqrt{2/\sigma\omega\mu_0}$, which is about 5% of the thickness $2b$. Then, taking the limit $A \to \infty$ in a distributional sense, we see that the current is restricted to the boundaries $z = \pm b$. Summarizing, we have taken the limits $\sqrt{\omega\epsilon_d/\sigma} \to 0$ and $A \to \infty$ to arrive at the asymptotic expressions for the fields. This corresponds to $\sigma \to \infty$, *i.e.*, the stripline is considered as a perfect conductor. Moreover, since the thickness $2b$ is much smaller than all other length scales, the stripline can be modeled as infinitely thin.

Applying the thus deduced field expressions as boundary conditions, we model a stripline in a stripline environment as a perfect and infinitely thin conductor. The propagating wave is assumed of TEM type. It is shown that the stripline can only support such a wave, if the wave number of the stripline equals the wave number of the dielectric layer. This is in correspondence with the aforementioned asymptotic expressions. A boundary-value problem for the electric potential is deduced, in which the total amplitude of the current is related to the potential difference between stripline and groundplates. The boundary-value problem is solved by two approaches, one leading to an integral equation for the current, and the other leading to an integral equation for an auxiliary function. A relation between the current and the auxiliary function is used to obtain asymptotic expressions of current and impedance. For $\zeta \ll 1$, the current is almost uniform and the asymptotic expression for the impedance matches with the numerical solution for $\zeta \lesssim 0{\cdot}2$. For $\zeta \gg 1$, the current is dominated by the square-root singular behaviour of the current near the edges of the stripline, and the asymptotic expression for the impedance matches with the numerical solution for $\zeta \gtrsim 3$. Furthermore, it is shown that the numerical solution for the impedance matches with results in the literature.

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